8.4 Digital Communication over Unreliable Channels

Definition 8.22. A simple digital communication channel called **binary symmetric channel** (BSC) is shown in Figure 49. This channel can be described as a channel that introduces random bit errors with probability p. This p is called the crossover probability.

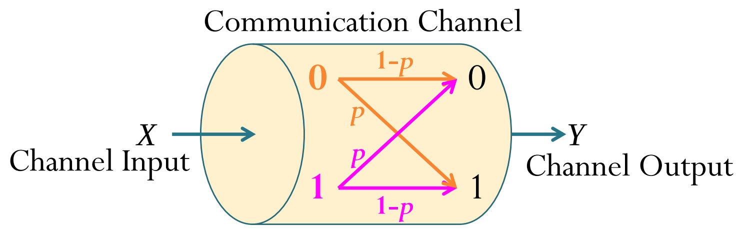

Figure 49: Binary Symmetric Channel (BSC)

Example 8.23. Error Correction Control via Repetition Code: Consider a digital communication system through the **binary symmetric channel** (BSC) discussed in Definition 8.22.

A crude digital communication system would put binary information into the channel directly; the receiver then takes whatever value that shows up at the channel output as what the sender transmitted. Such communication system would directly suffer bit error probability of p.

In situation where this error rate is not acceptable, error control techniques are introduced to reduce the error rate in the delivered information.

One method of reducing the error rate is to use error-correcting codes:

A simple error-correcting code is the ***repetition code***. Example of such code is described below:

- At the transmitter, the "encoder" box performs the following task:
  - To send a 1, it will send 11111 through the channel.
  - To send a 0, it will send 00000 through the channel.

- When the five bits pass through the channel, it may be corrupted. Assume that the channel is binary symmetric and that it acts on each of the bit independently.
- At the receiver, we (or more specifically, the decoder box) get 5 bits, but some of the bits may be changed by the channel.

  To determine what was sent from the transmitter, the receiver apply the ***majority rule***: Among the 5 received bits,

  - if #1 > #0, then it claims that "1" was transmitted,
  - if #0 > #1, then it claims that "0" was transmitted.

Two ways to calculate the probability of error:

(a) (transmission) error occurs if and only if the number of bits in error are ≥ 3.

(b) (transmission) error occurs if and only if the number of bits *not* in error are ≤ 2.

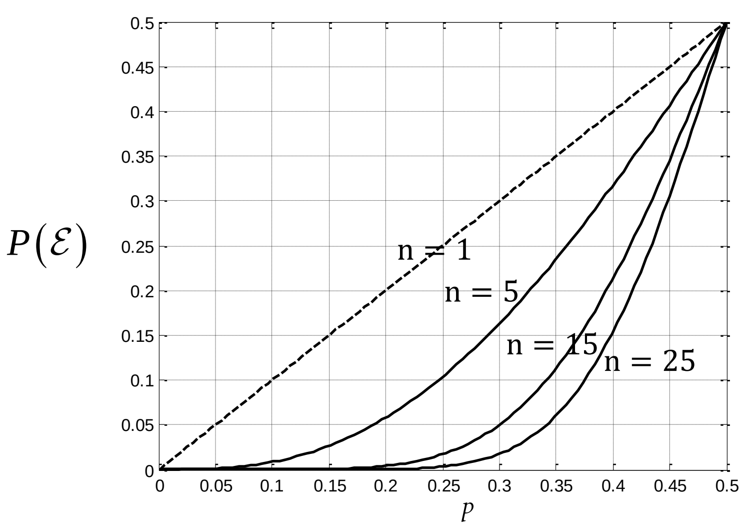

Figure 50: Bit error probability for a simple system that uses repetition code at the transmitter (repeat each bit n times) and majority vote at the receiver. The channel is assumed to be binary symmetric with bit error probability p.

Example 8.24 (Majority Voting in Digital Communication)**.** A certain binary communication system has a bit-error rate of 0.1; i.e., in transmitting a single bit, the probability of receiving the bit in error is 0.1. To transmit messages, a three-bit repetition code is used. In other words, to send the message 1, a "codeword" 111 is transmitted, and to send the message 0, a "codeword" 000 is transmitted. At the receiver, if two or more 1s are received, the decoder decides that message 1 was sent; otherwise, i.e., if two or more zeros are received, it decides that message 0 was sent.

Assuming bit errors occur independently, find the probability that the decoder puts out the wrong message.

[Gubner, 2006, Q2.62]

Solution: Let $p = 0.1$ be the bit error rate. Let $\mathcal{E}$ be the error event. (This is the event that the decoded bit value is not the same as the transmitted bit value.) Because majority voting is used, event $\mathcal{E}$ occurs if and only if there are at least two bit errors. Therefore

$$P(\mathcal{E}) = \binom{3}{2}p^2(1-p) + \binom{3}{3}p^3 = p^2(3-2p).$$

When $p = 0.1$, we have $P(\mathcal{E}) \approx \boxed{0.028}$.

Example 8.25. Consider a transmission over a binary symmetric channel (BSC) with crossover probability p. The random (binary) input to the BSC is denoted by X. Let p_1 be the probability that $X = 1$. (This implies the probability that $X = 0$ is $1 - p_1$.) Let Y by the output of the BSC.

(a) Suppose, at the receiver (which observes the output of the BSC), we learned that $Y = 1$. For each of the following scenarios, which event is more likely, "$X = 1$ was transmitted" or "$X = 0$ was transmitted"? (Hint: Use Bayes' theorem.)

  (i) Assume $p = 0.3$ and $p_1 = 0.1$.

  (ii) Assume $p = 0.3$ and $p_1 = 0.5$.

  (iii) Assume $p = 0.3$ and $p_1 = 0.9$.

  (iv) Assume $p = 0.7$ and $p_1 = 0.5$.

(b) Suppose, at the receiver (which observes the output of the BSC), we learned that $Y = 0$. For each of the following scenarios, which event is more likely, "$X = 1$ was transmitted" or "$X = 0$ was transmitted"?

(i) Assume $p = 0.3$ and $p_1 = 0.1$

(ii) Assume $p = 0.3$ and $p_1 = 0.5$

(iii) Assume $p = 0.3$ and $p_1 = 0.9$

(iv) Assume $p = 0.7$ and $p_1 = 0.5$

Remark: A MAP (maximum a posteriori) detector is a detector that takes the observed value Y and then calculate the most likely transmitted value. More specifically,

$$\hat{x}_{MAP}(y) = \arg\max_x P[X = x|Y = y]$$

In fact, in part (a), each of your answers is $\hat{x}_{MAP}(1)$ and in part (b), each of your answers is $\hat{x}_{MAP}(0)$.

Solution: First, recall that, in class, we define $P[X = x]$ to be $P([X = x])$. Here, we extend the such definition to conditional probability. In particular,

$$P[Y = y\,|X = x] = P([Y = y]\,|[X = x])\,.$$

Here, we are given that $P[X = 1] = p_1$. Applying $P(A^c) = 1 - P(A)$, we have $P[X = 0] = 1 - p_1$. We are also given that

$$P[Y = 1\,|X = 0] = P[Y = 0\,|X = 1] = p.$$

Applying $P(A^c\,|B) = 1 - P(A\,|B)$, we have

$$P[Y = 0\,|X = 0] = P[Y = 1\,|X = 1] = 1 - p.$$

(a) Here, we know that $Y = 1$. To find out what was transmitted, we compare $P[X = 0\,|Y = 1]$ and $P[X = 1\,|Y = 1]$. By Bayes' theorem,

$$P[X = 0\,|Y = 1] = \frac{P[Y = 1\,|X = 0]\,P[X = 0]}{P[Y = 1]} = \frac{p\,(1 - p_1)}{P[Y = 1]} = \frac{p - pp_1}{P[Y = 1]}$$

and

$$P[X = 1\,|Y = 1] = \frac{P[Y = 1\,|X = 1]\,P[X = 1]}{P[Y = 1]} = \frac{(1 - p)\,p_1}{P[Y = 1]} = \frac{p_1 - pp_1}{P[Y = 1]}$$

Note that both terms have "$-pp_1$" in the numerator and "$P[Y = 1]$" the denominator. So, we can simply compare the "p" and "p_1" parts.

(i) When $p = 0.3$ and $p_1 = 0.1$, we have $p > p_1$. Therefore, $P[X = 0\,|Y = 1] > P[X = 1\,|Y = 1]$. In other words, conditioned on $Y = 1$, the event $X = \boxed{0}$ is more likely.

(ii) When $p = 0.3$ and $p_1 = 0.5$, we have $p < p_1$. Therefore, $P[X = 0\,|Y = 1] < P[X = 1\,|Y = 1]$. In other words, conditioned on $Y = 1$, the event $X = \boxed{1}$ is more likely.

(iii) When $p = 0.3$ and $p_1 = 0.9$, we have $p < p_1$. Therefore, $P[X = 0 | Y = 1] < P[X = 1 | Y = 1]$. In other words, conditioned on $Y = 1$, the event $X = \boxed{1}$ is more likely.

(iv) When $p = 0.7$ and $p_1 = 0.5$, we have $p > p_1$. Therefore, $P[X = 0 | Y = 1] > P[X = 1 | Y = 1]$. In other words, conditioned on $Y = 1$, the event $X = \boxed{0}$ is more likely.

(b) In this part, we know that $Y = 0$. To find out what was transmitted, we compare $P[X = 0 | Y = 0]$ and $P[X = 1 | Y = 0]$. By Bayes' theorem,

$$P[X = 0 | Y = 0] = \frac{P[Y = 0 | X = 0] P[X = 0]}{P[Y = 0]} = \frac{(1 - p)(1 - p_1)}{P[Y = 0]} = \frac{1 - p - p_1 + pp_1}{P[Y = 0]}$$

and

$$P[X = 1 | Y = 0] = \frac{P[Y = 0 | X = 1] P[X = 1]}{P[Y = 0]} = \frac{pp_1}{P[Y = 0]} = \frac{pp_1}{P[Y = 0]}$$

Note that both terms have "$-pp_1$" in the numerator and "$P[Y = 0]$" the denominator. So, we can simply compare the "$1 - p - p_1$" and "''0 parts.

(i) When $p = 0.3$ and $p_1 = 0.1$, we have $1-p-p_1 = 0.6 > 0$. Therefore, $P[X = 0 | Y = 0] > P[X = 1 | Y = 0]$. In other words, conditioned on $Y = 0$, the event $X = \boxed{0}$ is more likely.

(ii) When $p = 0.3$ and $p_1 = 0.5$, we have $1-p-p_1 = 0.2 > 0$. Therefore, $P[X = 0 | Y = 0] > P[X = 1 | Y = 0]$. In other words, conditioned on $Y = 0$, the event $X = \boxed{0}$ is more likely.

(iii) When $p = 0.3$ and $p_1 = 0.9$, we have $1-p-p_1 = -0.2 < 0$. Therefore, $P[X = 0 | Y = 0] > P[X = 1 | Y = 0]$. In other words, conditioned on $Y = 1$, the event $X = \boxed{1}$ is more likely.

(iv) When $p = 0.7$ and $p_1 = 0.5$,we have $1-p-p_1 = -0.2 < 0$. Therefore, $P[X = 0 | Y = 0] > P[X = 1 | Y = 0]$. In other words, conditioned on $Y = 0$, the event $X = \boxed{1}$ is more likely.